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CONCERNING ELASTIC AND PLASTIC COMPONENTS OF DEFORMATION

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Abstract—The common assumption of defining elastic and plastic strain increments about a stressed configuration by adding and removing an infinitesimal increment of stress, $\Delta\sigma$ and recording the resulting reversible and residual strain increments is examined. It is shown that part of the residual strain arises from rotation of the body, because, throughout the process it is subject to the stress σ. This component of the residual strain is associated only with elastic deformation and is incorrectly ascribed to plastic flow. Definition of plastic flow as permanent deformation after all macroscopic stresses are removed eliminates the anomaly and provides a sound theory for materials which do not exhibit plastic flow on unloading.

1. INTRODUCTION

For infinitesimal deformation theory in which displacements are small so that the difference between the deformed and undeformed geometries of a body can be neglected, strain components are given by the linear relation

$$\epsilon_{ij} = (u_{i,j} + u_{j,i})/2 \tag{1.1}$$

with the usual notation of u_i being displacement components and $,j$ denoting the operator $\partial/\partial x_j$, x_j being Cartesian position coordinates. The plastic strain tensor, ϵ^p, expressing permanent change of shape and the reversible elastic strain, ϵ^e, then give the total strain by addition

$$\epsilon = \epsilon^p + \epsilon^e. \tag{1.2}$$

Differentiation with respect to time then gives the addition law for strain-rates

$$\dot{\epsilon} = \dot{\epsilon}^p + \dot{\epsilon}^e. \tag{1.3}$$

For finite deformations Fig. 1 depicts the configuration (x) of a body subjected to plastic and

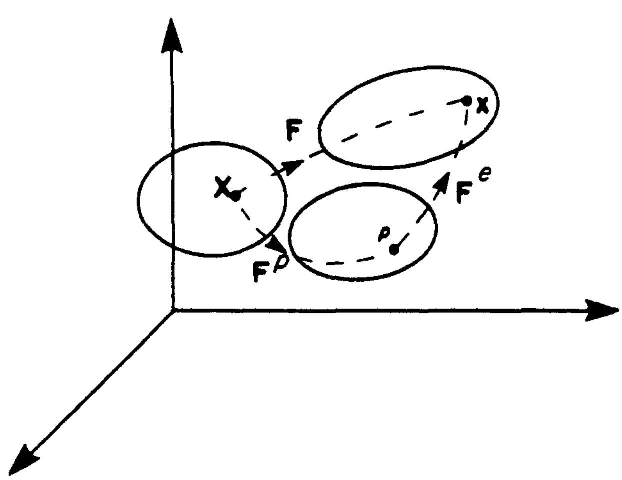

Fig. 1. Total elastic–plastic deformation expressed in plastic and elastic components.

elastic deformation from its undisturbed reference state (X). In the unstressed configuration (p) the stresses have been removed and complete recovery of the elastic strains with no further plastic flow is considered to have taken place. The deformation gradient

$$F_{ij} = \partial x_i / \partial X_j \tag{1.4}$$

expresses the deformation for the continuous differentiable mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. Since non-homogeneous elastic-plastic deformation usually leaves residual stresses in an unloaded specimen, removal of stress can only be achieved in the limit when the body is dissected into vanishingly small elements [1–3]. The elastic deformation associated with the residual stresses is needed to provide a continuous mapping to the unloaded configuration and the unstressed elements will usually not constitute a continuous, one to one, mapping of the body.

As an example, Fig. 2 shows a hollow circular cylinder subjected to internal pressure in plane-strain until the plastic region has spread to an intermediate radius c. The original configuration of the elastic region before pressure had been applied could represent the unstressed stated

$$\mathbf{p}(\mathbf{X}, t) = \mathbf{X} \tag{1.5}$$

since no plastic deformation has occurred there. However, the inner rings of material subject to plastic flow have increased in radius and would over-lap the material subjected to only elastic deformation. This would correspond to a mapping function prescribing the same unstressed radius for points initially inside and outside the region which became plastic. Alternatively such interpenetration could be avoided by slitting the outer material which had been subject to elastic deformation only, on radial planes permitting the sectors to move out radially, but this would generate gaps between the sectors and hence a discontinuous mapping. Thus deformation gradients $\mathbf{F}^p = \partial \mathbf{p} / \partial \mathbf{X}$ (and also $\mathbf{F}^e = \partial \mathbf{x} / \partial \mathbf{p}$) will not exist but can be replaced by local linear mapping matrices, $\mathbf{F}^p(\mathbf{X}, t)$ and $\mathbf{F}^e(\mathbf{X}, t)$, associated with infinitesimal neighborhoods of each material point, as discussed by Nemat-Nasser [4]. The mapping sequence relation [3]

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \tag{1.6}$$

which expresses the gradient of the total deformation in terms of the elastic and plastic components, remains valid.

2. CONCEPTS

The onset of plastic flow is detected by loading a specimen, and then removing the load to relieve the elastic strain so that plastic strain, the permanent change in dimensions of the

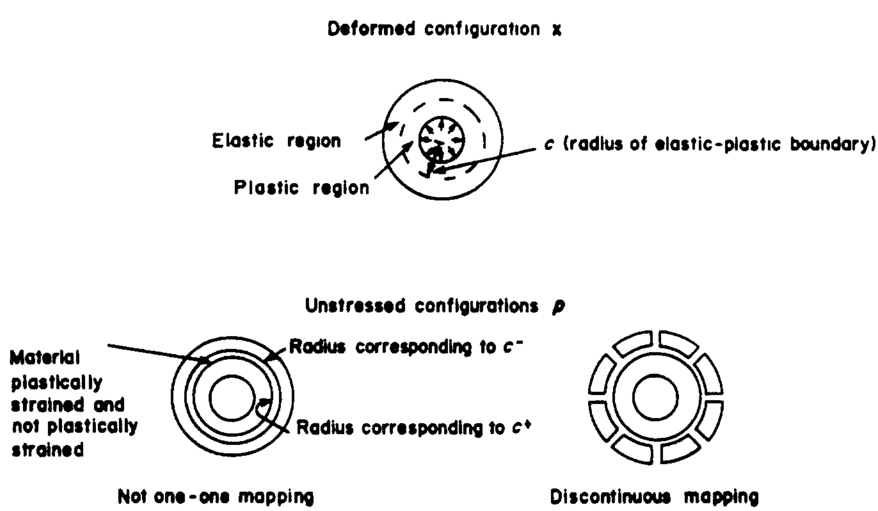

Fig. 2. Unstressed states following partially plastic loading of a cylinder.

unstressed body, becomes evident. The accuracy with which the residual strain is measured is termed the offset strain. Thus a strain change measured at zero macroscopic stress constitutes plastic deformation. This permanent shape change is associated with the passage of dislocations through the crystals of the metal and other permanent re-arrangements of the atomic lattice. This concept leads to the definition of the plastic deformation gradient, $\mathbf{F}^p$, illustrated in Fig. 1, associated with the change in shape following elastic-plastic deformation after the stress and hence the elastic strain are removed. If stress is then re-applied, initially elastic deformation of the unstressed or current natural configuration is produced. It has been found experimentally that the elastic constants of a metal are not appreciably influenced by plastic flow at moderate strains [5, 6]. At large plastic strains anisotropy can be induced and the elastic characteristics are modified by the plastic flow. Hill [7] gives the plastic strain at which this effect becomes significant at about 30%. Thus in the moderate strain range the usual elastic laws which apply for an initially undeformed metal can be applied to the elastic deformation of the unstressed, plastically deformed configuration defined by the coordinates p in Fig. 1. These considerations constitute the basis for the selection of the variables in Fig. 1 and for the expression for the stress in the elastically-plastically deformed configuration, x in terms of the elastic deformation gradient $\mathbf{F}^e$ and the usual elastic law. Note that the history of elastic deformation $\mathbf{F}^e(\mathbf{X}, t)$ in the elastic-plastic case is exactly the same as purely elastic deformation from an unstressed reference state with the same stress history if incompatibility constraints do not arise. This formulation is consistent with the physical basis of plastic flow since even after appreciable plastic strain only a small proportion of the atoms are disturbed from the regular atomic lattice and the atomic lattice determines the elastic constants of the material. For randomly oriented crystallites a poly-crystalline material is macroscopically isotropic, and this condition is not significantly changed after moderate strain.

3. STRAIN INCREMENTS ABOUT THE DEFORMED STATE

It is often stated, or assumed, that the elastic and plastic components of strain increment, or equivalently of strain rate, can be satisfactorily defined by considering the reversible and residual components of infinitesimal strain increments about the current stressed configuration when a small increment of stress $\Delta\sigma$ is added and then removed (see, e.g. Nemat-Nasser [4] who cites Hill [8]†). In rate terms an increment of strain is equivalent to considering velocity strain (the symmetric part of the velocity gradient, sometimes termed deformation rate) multiplied by the time increment Δt. The increment of elastic strain is defined to be the reversible strain increment associated with the stress increment and the plastic increment of strain is defined to be the residual increment. Thus

$$\Delta\epsilon = \Delta\epsilon^e + \Delta\epsilon^p \tag{3.1}$$

or

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p. \tag{3.2}$$

The velocity strain can be obtained from (1.6) by evaluating the symmetric part of the velocity gradient in the deformed state x. Since

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t}\bigg|_{\mathbf{X}} = \mathbf{v}. \tag{3.3}$$

is the velocity of a particle of the continuum, then the velocity gradient is:

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\,\mathbf{F}^{-1} \tag{3.4}$$

where $\dot{\mathbf{F}} = \partial \mathbf{F}/\partial t|_{\mathbf{X}}$. In terms of the elastic and plastic local deformation gradients $\mathbf{F}^e$ and $\mathbf{F}^p$

†This citation is not strongly definitive since Hill merely states, "For a plastic element the simplest hypothesis is ... ".

already discussed, material time differentiation of (1.6), that is at fixed $\mathbf{X}$, yields

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^e \dot{\mathbf{F}}^p \tag{3.5}$$

so that the velocity gradient (3.4) becomes

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{F}}^e\mathbf{F}^{e^{-1}} + \mathbf{F}^e\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}\mathbf{F}^{e^{-1}}. \tag{3.6}$$

The purely plastic deformation rate in the configuration $\mathbf{p}$ in Fig. 1 is given by the symmetric part of $\mathbf{L}^p$

$$\mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}. \tag{3.7}$$

Since $\mathbf{F}^e(t)$ is the elastic deformation gradient, the rate of elastic strain about the current configuration is

$$\mathbf{D}^e = (\dot{\mathbf{F}}^e\mathbf{F}^{e^{-1}})_S \tag{3.8}$$

where the subscript S denotes the symmetric part of the matrix in parentheses. This usually cannot be directly expressed in terms of a velocity gradient since, due to plastic flow, the unstressed reference state needed for application of the usual elastic law is deforming in a space-wise discontinuous and non-differentiable manner.

Thus with the variables appropriate for elastic and plastic deformation the sum of the elastic and plastic deformation rates taken from (3.8) and (3.7) respectively, is not equal to the total deformation rate given by the symmetric part of (3.6). However, in elastic-plastic bodies deviatoric stress components are limited by the yield condition to magnitudes which correspond to elastic strains of the order 10^{-3}: (yield stress/elastic modulus) and free surfaces or surfaces on which traction is prescribed commonly limit all elastic strains to be of this order. Without loss of generality, rotation can be embodied in $\mathbf{F}^p$ [3] so that $\mathbf{F}^e$ becomes a symmetric pure deformation matrix which takes the form

$$\mathbf{F}^e \approx \mathbf{I} + \boldsymbol{\delta} \tag{3.9}$$

where $\boldsymbol{\delta}$ is of the order 10^{-3}. Thus the last term of (3.6) is closely approximated by (3.7) and to this accuracy the total strain rate is equal to the sum of the elastic and plastic strain rates using the definitions already given and the velocity strain to represent the strain rate about the current configuration. The exception for which plastic flow cannot limit elastic strains is loading in high, predominantly hydrostatic compression as often occurs in shock waves [9, 10]. Then (3.9) takes the form

$$\mathbf{F}^e = \alpha^{1/3}(\mathbf{I} + \boldsymbol{\delta}) \tag{3.10}$$

where $\alpha < 1$ is the volume compression ratio. The α cancels in the last term of (3.6), so that again this term is closely approximated by (3.7) and summation of appropriately defined elastic and plastic strain-rates pertains to a high degree of approximation. Thus, an adequate theory can be based on:

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \tag{3.11}$$

$\mathbf{D}$ and $\mathbf{D}^p$ being the symmetric parts of $\mathbf{L}$ and $\mathbf{L}^p$ given by the first part of (3.6) and (3.7) respectively. This is the basis for much work in elastic–plastic analysis, such as the finite element program [11] and the metal forming investigations [12] and [13].

While as stated in the opening paragraph of this section, summation of strain rates, (3.2), is commonly adopted, consideration seems usually to be limited to analyses of an increment of deformation. Integration of strain increments to determine total plastic strain seems not to be studied. For finite element evaluation [11–13] such a study is by-passed since evaluation of the solution is treated by means of a Lagrangian type finite element mesh which is convected and deforms with the body and hence exhibits the resultant deformation. This could be partitioned into elastic and plastic components by analysing the destressing process. Usually the final deformation and stress distribution, including the residual stresses generated by heterogeneous plastic flow, are the quantities of interest.

In view of the approximation involved in the eqn (3.11), it seems appropriate to examine the significance of the terms in (3.6) particularly with regard to the nature of the elastic and plastic contributions. As we have already seen, (3.8) gives the rate of elastic deformation which corresponds to the velocity strain in an elastic body with a fixed reference state and the same stress history. Because elasticity is an instantaneous relation between stress and deformation, deformation functions such as $\mathbf{F}^e$ involve only the instantaneous configuration of the unstressed reference state, and hence can be defined and differentiated with respect to time directly without separate analysis of the motions of the reference and stressed configurations.

As is currently adopted for most evaluations, we will consider isotropic material response in both elastic and plastic deformation, since this is sufficient to bring out the significance of the last term in (3.6) which embodies the plasticity contribution to the deformation rate or velocity strain. This last term can be written:

$$\mathbf{F}^e(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_S\mathbf{F}^{e^{-1}}+\mathbf{F}^e(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_A\mathbf{F}^{e^{-1}} \tag{3.12}$$

where S denotes the symmetric part and A the anti-symmetric part.

Now the terms in the relation (1.6) are not uniquely determined since without loss of generality, rotation can be associated with either $\mathbf{F}^e$ or $\mathbf{F}^p$, or both, as already mentioned, and discussed in [3]. It is convenient to consider elastic unloading from $\mathbf{x}$ to $\mathbf{p}$ in Fig. 1 without rotation so that $\mathbf{F}^e$ will be a symmetric matrix. For an isotropic material $\mathbf{F}^e$ will have principal axes parallel to those of the stress tensor $\mathbf{T}$, as also will the plastic velocity strain $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_S$. Thus the three matrices in the first term of (3.12) all have the same principal axes, the multiplication is therefore commutative and the first and last factors cancel. Thus the first term in (3.12) reduces to the plastic velocity strain, the symmetric part of (3.7). Incidentally, only this deformation rate and, for isotropic hardening, the current tensile yield strength appear in the constitutive relation as effects of plasticity. The detail of the history of plastic deformation, $\mathbf{F}^p(\mathbf{X}, t)$ appears only in the scalar work-hardening variable.

The anti-symmetric matrix in the second term in (3.12) expresses the rate of rotation in the plastically deformed configuration p. The deformation rate component defined by the second operator in (3.12) comprises a sequence of three motions. The final post-multiplier $\mathbf{F}^{e^{-1}}$ which operates first represents the elastic deformation due to de-stressing. Then follow a rotation in the increment of time dt after which the elastic strain is re-applied. This sequence results in a change in deformation equal to the difference in elastic deformations due to stress $\boldsymbol{\sigma}$ applied in incrementally rotated directions relative to the body. This will appear as a residual strain when the stress-increment $\Delta\boldsymbol{\sigma}$ is removed, since the initial stress $\boldsymbol{\sigma}$ will still be retained. According to the procedure expressed in (3.2), the residual strain (not reversible with $\Delta\boldsymbol{\sigma}$) will be allocated to the plastic strain increment. Since it is due to differing elastic strains caused by incremental rotation of the plastic configuration, this allocation is clearly incorrect. Yet it arises by ascribing residual strains, after applying and removing the stress increments $\Delta\sigma$, to plastic flow, the elastic component being the reversible strain increment.

Technically this residual strain arises because the elastic coupling matrices in the last term in (3.12) cancel out the antisymmetric property of the plastic rotation matrix and hence yield a residual strain.

It is interesting to note that no contribution to plastic work arises from the second term in (3.12) as shown in [3]. Pre-multiplying by the stress tensor and taking the trace permits cyclic permutation of $\mathbf{F}^{e^{-1}}$ ahead of the stress, and then cancellation with $\mathbf{F}^e$ since $\mathbf{F}^e$ and T have common principal axes. Then the stress $\mathbf{T}$ does no work on the anti-symmetric rotation deformation, although this term does contribute to the residual strain.

The use of the Jaumann derivative of stress for the elastic-plastic constitutive relation in the development of the computer program presented by McMeeking and Rice[11], goes beyond infinitesimal theory by incorporating rotation effects. It can be considered equivalent to utilizing axes which have the same spin as the body. The velocity strain or rate of deformation is not affected by the axes rotation, but the spin relative to these axes is reduced to zero. In the common circumstance of dominating plastic-strain increments, zero spin has the effect of reducing the magnitude of the spin coupling term, the second term in (3.12), which appears in the complete non-linear theory. This is so since with dominating plastic-strain increments, the plastic spin

$$\mathbf{W}^p = (\dot{\mathbf{F}}^p \mathbf{F}^{p^{-1}})_A \tag{3.13}$$

is a close approximation to the total spin

$$\mathbf{W} = (\dot{\mathbf{F}}\mathbf{F}^{-1})_A. \tag{3.14}$$

Thus the term which constitutes the difference between the symmetrical parts of (3.6), and (3.11) is small and a theory based on the latter will constitute a close approximation to the complete non-linear theory based on (1.6). The incremental development based on (1.6) and the comparison with incremental theories based on (3.11), which includes virtually all computer codes for elastic–plastic analysis at finite strain, will be given in a forthcoming paper.

4. THE CASE OF PRINCIPAL DIRECTIONS FIXED IN THE BODY

When principal directions are fixed in the body, the analysis can be expressed with respect to principal axes and all matrices become diagonal[9]. If λ_i $i = 1,2,3$ are the stretch ratios of the total deformation, λ_i^p the stretch ratios of the plastic deformation and λ_i^e of the elastic deformation,

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathbf{F}^p = \begin{pmatrix} \lambda_1{}^p & 0 & 0 \\ 0 & \lambda_2{}^p & 0 \\ 0 & 0 & \lambda_3{}^p \end{pmatrix} \mathbf{F}^e = \begin{pmatrix} \lambda_1{}^e & 0 & 0 \\ 0 & \lambda_2{}^e & 0 \\ 0 & 0 & \lambda_3{}^e \end{pmatrix} \tag{4.1}$$

Relation (1.6) becomes:

$$\lambda_i = \lambda_i^e \lambda_i^p,\ i \text{ not summed} \tag{4.2}$$

the matrix multiplication now being commutative. Natural or logarithmic strains can be defined as the natural logarithms of the stretch ratios:

$$\epsilon_i = \ln \lambda_i,\ \epsilon_i^e = \ln \lambda_i^e,\ \ \epsilon_i^p = \ln \lambda_i^p \tag{4.3}$$

and (4.2) becomes

$$\epsilon_i = \epsilon_i^e + \epsilon_i^p. \tag{4.4}$$

Strain rates are also additive since differentiating (4.2) with respect to time and dividing by λ_i yields

$$\dot{\lambda}_i/\lambda_i = \dot{\lambda}_i^e/\lambda_i^e + \dot{\lambda}_i^p/\lambda_i^p \tag{4.5}$$

hence or equivalently by differentiating (4.4)

$$\dot{\epsilon}_i = \dot{\epsilon}_i^e + \dot{\epsilon}_i^p \tag{4.6}$$

Thus with axes fixed in the body, and analysis based on these axes so that no rotation is involved, elastic and plastic rates of strain are additive (4.6), or equivalently increments of strain, so that reversible and residual strain increments due to application and removal of an increment of stress will constitute elastic and plastic strain increments. The residual plastic increment will express exactly the increment of permanent deformation measured at zero macroscopic stress, with no coupling to changes in elastic strain. With this simple configuration the plastic strain is expressed relative to the undeformed reference state as is the total strain, but the elastic strain is expressed relative to the unstressed configuration, the natural and most convenient form for formulating the elastic constitutive relation. Moreover elastic and plastic strains are also additive (4.4). With rotation, the analysis loses this simplicity because logarithmic functions of matrices do not satisfy the simple additive property associated with logarithms of scalars because commutativity is no longer valid.

5. DISCUSSION

We are concerned with the analysis of elastic and plastic increments of strain when the body is already stressed so that plastic strain is generated in material continuously strained elastically. Strain increments are measured without reducing the stress to zero in which condition plastic strain could be uncoupled from elastic strain.

Since the analysis of elastic and plastic increments of strain associated with application and removal of a stress increment $\Delta\boldsymbol{\sigma}$ to a body already subjected to stress $\boldsymbol{\sigma}$, takes on a simple form when the body does not rotate relative to the stress tensor $\boldsymbol{\sigma}$ and the stressing is homogeneous, such a deformation is most convenient for measuring elastic–plastic material properties. As already shown, the increment of residual strain about the current geometry then expresses the increment of plastic natural strain about the undeformed reference state—the permanent deformation exhibited at zero macroscopic stress, i.e. zero elastic strain. The increment of reversible strain expresses the increment of elastic natural strain about the plastically deformed unstressed state. Measurement of these quantities as the body is loaded will provide the elastic–plastic characteristics of the material.

As already discussed, since plastic strains are commonly large compared to elastic strains, the error associated with ascribing the residual strain increment, following an incremental pulse of stress, to plastic flow will commonly be small although such residual strain *can* be non-zero when plastic strain increments are zero. Thus, continued application of total strain rate (based on velocity strain in the current configuration) as the sum of elastic and plastic component is probably justified until more precise approaches are developed.

If tests including rotation are desired, the evaluation of plastic and elastic components of deformation gradients $\mathbf{F}^p$ and $\mathbf{F}^e$, and hence of finite strain, defined, e.g. by Lagrange strain $(\mathbf{F}^{p^T}\mathbf{F}^p - \mathbf{I})/2$, $(\mathbf{F}^{e^T}\mathbf{F}^e - \mathbf{I})/2$ rests on manipulation of (3.6). As in the previous discussion there is no loss of generality in associating the rotation with $\mathbf{F}^p$ so that $\mathbf{F}^e$ is a symmetric pure deformation matrix.

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